Enrollment No: ______________________ **Exam Seat No:** ______________________

C.U.SHAH UNIVERSITY

Summer Examination-2018

Subject Name: Number Theory

Subject Code: 5SC04NUT1 **Branch: M.Sc. (Mathematics)**

Semester: 4 **Date: 03/05/2018** **Time: 10:30 To 01:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the following questions** **(07)**

a. Find $\gcd(306, 657)$. **(02)**

b. If $ca \equiv cb\ (mod\ n)$ and $\gcd(c, n) = 1$, then prove that $a \equiv b\ (mod\ n)$. **(02)**

c. State fundamental theorem of arithmetic. **(01)**

d. Calculate: $\phi(360)$. **(01)**

e. If p is a prime and $p|ab$, then $p|a$ or $p|b$. Determine whether the statement is True or False. **(01)**

Q-2 **Attempt all questions** **(14)**

a. Define: Mobious function. Prove that Mobious function is multiplicative function. **(05)**

b. State and prove fundamental theorem of divisibility. **(05)**

c. Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square. **(04)**

OR

Q-2 **Attempt all questions** **(14)**

a. State and prove Euclidean algorithm. **(05)**

b. Let $N = a_0 + a_1 10 + a_2 10^2 + \cdots + a_m 10^m$ be the decimal expansion of the positive integer $N, 0 \le a_k < 10$, and let $S = a_0 + a_1 + \cdots + a_m$. Then prove that $9|N$ if and only if $9|S$. Is 1571724 divisible by 9? Justify. **(05)**

c. If p_n is the n^{th} prime numbers, then prove that $p_n < 2^{2^n}, \forall\ n$. **(04)**

Q-3 **Attempt all questions** **(14)**

a. State Chinese remainder theorem. Solve the system of three congruences $x \equiv 1(mod\ 3), x \equiv 2(mod\ 5), x \equiv 3(mod\ 7)$. **(05)**

b. Prove that for any choice of positive integers a and b, $lcm\ (a, b) = ab$ if and only if $\gcd(a, b) = 1$. **(05)**

c. Find highest power of 3 that divides 81!. **(02)**

d. Let x and y be real numbers. Then prove that **(02)**

$$[x] + [-x] = \begin{cases} 0, & if\ x\ is\ an\ integer \\ -1, & otherwise \end{cases}.$$

OR

Q-3 **Attempt all questions** **(14)**

a. If $2^k - 1$ is prime$(k > 1)$, then prove that $n = 2^{k-1}(2^k - 1)$ is perfect and every even perfect number is of the form. **(05)**

b. Solve: $9x \equiv 21\ (mod\ 30)$. **(05)**

c. Find the number of multiple of 11 among the integer 300to 1000. **(02)**

d. If p is prime and $k > 0$, then prove that $\phi(p^k) = p^k\left(1 - \frac{1}{p}\right)$. **(02)**

SECTION – II

Q-4 **Attempt the following questions** **(07)**

a. Compute the convergents of the simple continued fraction$[8; 1, 1,2]$. **(02)**

b. Express the rational number $\frac{19}{51}$ in finite simple continue fraction. **(02)**

c. Define: Algebraic number. **(01)**

d. Define: Index of a relative to r. **(01)**

e. State Wilson's theorem. **(01)**

Q-5 **Attempt all questions** **(14)**

a. Prove that if $c_k = \frac{p_k}{q_k}$ is the k^{th} convergent of the finite simple continued fraction$[a_0; a_1, a_2, \dots, a_n]$, then$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1},\ 1 \le k \le n.$ **(05)**

b. Determine the infinite continued fraction representation of irrational number$\sqrt{23}$. **(05)**

c. Solve the linear Diophantine equation $172x + 20y = 1000$. **(04)**

OR

Q-5 **Attempt all questions** **(14)**

a. Let x be an arbitrary irrational number. If the rational number $\frac{a}{b}$, where $b \ge 1$ and $\gcd(a, b) = 1$, satisfies$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$, then prove that $\frac{a}{b}$ is one of the convergent $\frac{p_n}{q_n}$ in the continued fraction representation of x. **(05)**

b. Find all relatively prime solution of the equation $x^2 + y^2 = z^2$with $0 < z < 30$. **(05)**

c. Find all primitive roots of 17. **(04)**

Q-6 **Attempt all questions** **(14)**

a. Let n be a positive rational integer and ξ a complex number. Suppose that the complex numbers $\theta_1, \theta_2, \theta_3, \dots, \theta_n$, not all zero, satisfy the equation **(05)**

$$\xi\theta_j = a_{j,1}\theta_1 + a_{j,2}\theta_2 + a_{j,n}\theta_n, \qquad j = 1,2,3, \dots$$

where the n^2 coefficients $a_{j,i}$are rational. Then prove that ξ is an algebraic number. Moreover, if the $a_{j,i}$ are rational integers, ξ ia an algebraic integer.

b. If $\frac{p_k}{q_k}$ are the convergents of the continuous fraction expansion of $\sqrt{d}$, then prove that $p_k^2 - dq_k^2 = (-1)^{k+1}t_{k+1}$ where $t_{k+1} > 0, k = 0,1, 2, \dots$ **(05)**

c. Prove that the norm of a product equals the product of the norms in $Q(\sqrt{m})$. **(04)**

OR

Q-6 **Attempt all questions** **(14)**

a. Prove that the product of two primitive polynomial is primitive. **(05)**

b. If p is prime and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_n$is incongruent to 0 modulo p, is a polynomial of degree $n \ge 1$ with integral coefficients, then prove that $f(x) \equiv 0(mod\ p)$ has at most n incongruent solutions modulo p. **(05)**

c. Prove that if an irreducible polynomial $p(x)$ divides a product $f(x)g(x)$, then $p(x)$ divides at least one of the polynomial $f(x)$ and $g(x)$. **(04)**